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Box and ball system as a realization of ultradiscrete nonautonomous KP equation

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Abstract. Cellular automata, which are realized by dynamics of several kinds of balls in an infinite array of boxes, are investigated. They show soliton patterns even in the case when each box has arbitrary capacity. The analytical expression for the soliton patterns are obtained using ultradiscretization of the nonautonomous discrete KP equation.

1. Introduction

Cellular automata (CAs) serve as simple models for complex phenomena such as pattern formation, chaos and fractals [1]. They also exhibit coherent structures as is seen in the game of life [2]. Patterns which behave like solitons are also observed and discussed in several CA systems [3–5]. About a decade ago, one of the authors (DT) and Satsuma proposed a 1(space) +1(time) dimensional CA in which all patterns look like *solitons* analogous to that of the soliton solutions in nonlinear partial differential equations [6]. The CA takes a value of either zero or one. The rule to determine the value of the CA at position n and time $t+1$, x_n^{t+1}, is given as

$$x_n^{t+1} = \begin{cases} 1 & \text{if} \quad x_n^t = 0 \quad \text{and} \quad \sum_{n'=-\infty}^{n-1} x_{n'}^t > \sum_{n'=-\infty}^{n-1} x_{n'}^{t+1} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

Here we assume that the number of '1' is finite, that is, we take $\lim_{|n|\to\infty} x_n^t = 0$ as the boundary condition. An example of soliton patterns is shown in figure 1.

Soon after this proposal of the CA, DT extended it to so-called box and ball systems (BBSs) [7]. The idea is to consider x_n^t as the number of balls in the nth box at time t. Then the CA is represented as a system with an infinite array of boxes each of which is either empty or contains a ball. The evolution rule from t to $t+1$ is described as

(1) Move every ball only once.
(2) Move the leftmost ball to the nearest right empty box.
(3) Move the leftmost ball among the rest to its nearest right empty box.
(4) Repeat this procedure until all of the balls are moved.

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$t = 0$ ...11000100000000...
$t = 1$ ...00110010000000...
$t = 2$ ...00001101000000...
$t = 3$ ...00000010110000...
$t = 4$ ...00000001001100...
$t = 5$ ...00000000100011...

Figure 1. Two-soliton interactions of the soliton CA.

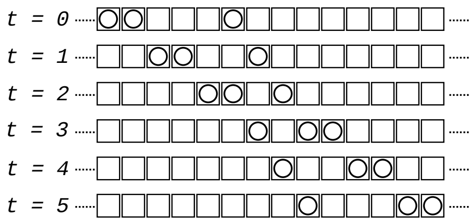

Figure 2. BBS corresponding to figure 1.

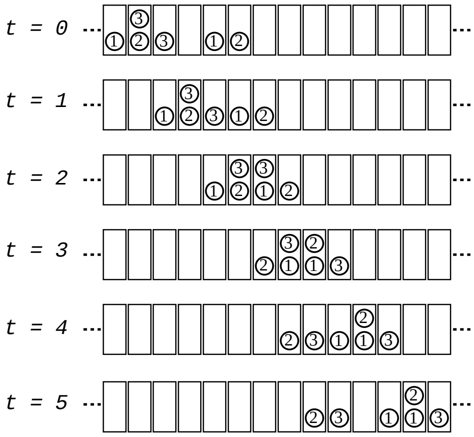

Figure 3. Two-soliton interaction of an extended BBS. $L = 2$ and $M = 3$.

We can easily see that this rule is equivalent to that of the original CA. Figure 2 shows the BBS corresponding to figure 1.

With this interpretation, we can introduce two extra freedoms: capacity of boxes and species of the ball. We suppose that the capacity of the box is L and there are M kinds of balls which are indexed by integers $1, 2, \ldots, M$. Then, the natural rule from t to $t + 1$ for the dynamics of the BBS would be

(1) Move every ball only once.
(2) Move the leftmost ball with index 1 to the nearest right box with space, i.e., to the nearest right box which contains less than L balls.
(3) Move the leftmost ball with index 1 among the rest to its nearest right box with space.
(4) Repeat this procedure until all of the balls with index 1 are moved.
(5) Do the same procedure (2)–(4) for the balls with index 2.
(6) Repeat this procedure successively until all of the balls are moved.

Surprisingly, the patterns of the BBSs also behave like solitons [7]. We show an example in figure 3.

Several years ago, the authors and Satsuma found a direct link between the BBS (1) and the soliton equations [9]. They showed a method by which CAs are obtained from continuous equations. This method is based on limiting procedures and is called ultra-discretization (UD) [10]†. In this paper, we will investigate the BBS, allowing that the capacities of the boxes

† This name was given by B Grammaticos.

differ in position, in terms of UD of the nonautonomous discrete KP (NDKP) equation [11,12]. The expressions of the soliton patterns are given through UD of the soliton solutions of the NDKP equation.

2. The NDKP equation

In the theory of KP hierarchy (Sato theory), the generating formula for a series of equations of the hierarchy is given by [13, 14]

$$\mathrm{Res}_{\lambda=\infty}\left[\tau\left(\boldsymbol{t}+\epsilon\left(\frac{1}{\lambda}\right)\right)\tau\left(\boldsymbol{t}'-\epsilon\left(\frac{1}{\lambda}\right)\right)\mathrm{e}^{\xi(\boldsymbol{t}-\boldsymbol{t}',\lambda)}\right]=0 \qquad (2)$$

where $\boldsymbol{t} = (t_1, t_2, t_3, \ldots)$ denotes an infinite number of independent variables, $\epsilon(\frac{1}{\lambda}) = (1/\lambda, 1/(2\lambda^2), 1/(3\lambda^3), \ldots)$, and $\xi(\boldsymbol{t}, \lambda) = \sum_{j=1}^{\infty} t_j\lambda^j$. One of the main results in the Sato theory is that a function τ satisfies equation (2) if and only if it corresponds to a GL_∞-orbit of the fermion vacuum (a highest weight vector in basic representation of GL_∞). Its coordinates are given through boson–fermion correspondence, and we can obtain the explicit expression of function τ. From equation (2), we have the so-called Fay identity for τ:

$$\begin{aligned}(b-c)\tau\left(\boldsymbol{t}-\epsilon\left(\frac{1}{a}\right)\right)\tau\left(\boldsymbol{t}-\epsilon\left(\frac{1}{b}\right)-\epsilon\left(\frac{1}{c}\right)\right)+(c-a)\tau\left(\boldsymbol{t}-\epsilon\left(\frac{1}{b}\right)\right)\tau\left(\boldsymbol{t}-\epsilon\left(\frac{1}{c}\right)\right.\\ \left.-\epsilon\left(\frac{1}{a}\right)\right)+(a-b)\tau\left(\boldsymbol{t}-\epsilon\left(\frac{1}{c}\right)\right)\tau\left(\boldsymbol{t}-\epsilon\left(\frac{1}{a}\right)-\epsilon\left(\frac{1}{b}\right)\right)=0. \qquad (3)\end{aligned}$$

Noticing that this identity resembles the discrete analogue of generalized Toda equation proposed by Hirota [15], Miwa found transformations which map the generating formula to discrete bilinear equations [16]. For example, by setting $\boldsymbol{t} = \ell\epsilon(\frac{1}{a}) + m\epsilon(\frac{1}{b}) + n\epsilon(\frac{1}{c})$ and $\tau(\ell, m, n) \equiv \tau(\boldsymbol{t})$, we have the discrete KP equation (Hirota–Miwa equation), which produces many important discrete integrable nonlinear equations [15]. The NDKP equation is obtained from the Fay identity by setting

$$\boldsymbol{t}=\sum_{\ell'}^{\ell}\epsilon\left(\frac{1}{a_{\ell'}}\right)+\sum_{m'}^{m}\epsilon\left(\frac{1}{b_{m'}}\right)+\sum_{n'}^{n}\epsilon\left(\frac{1}{c_{n'}}\right)$$

where

$$\sum_{k'}^{k}\equiv\begin{cases}\displaystyle\sum_{k'=1}^{k} & k\geqslant 1\\ 0 & k=0\\ \displaystyle-\sum_{k'=k+1}^{0} & k\leqslant -1.\end{cases}$$

Then, $\tau(\ell, m, n) \equiv \tau(\boldsymbol{t})$ satisfies

$$\begin{aligned}(b_m-c_n)\tau(\ell-1,m,n)\tau(\ell,m-1,n-1)+(c_n-a_\ell)\tau(\ell,m-1,n)\tau(\ell-1,m,n-1)\\ +(a_\ell-b_m)\tau(\ell,m,n-1)\tau(\ell-1,m-1,n)=0. \qquad (4)\end{aligned}$$

This equation is the NDKP equation. Taking $a_\ell = 0,\ b_m = 1,\ c_n = 1 + \delta_n$, equation (4) turns into

$$\begin{aligned}-\delta_n\tau(\ell-1,m,n)\tau(\ell,m-1,n-1)+(1+\delta_n)\tau(\ell,m-1,n)\tau(\ell-1,m,n-1)\\ -\tau(\ell,m,n-1)\tau(\ell-1,m-1,n)=0. \qquad (5)\end{aligned}$$

The N-soliton solution to equation (5) is given by [11, 17]

$$\tau(\boldsymbol{t}) = \langle \mathrm{vac}|g(\boldsymbol{t})|\mathrm{vac}\rangle \tag{6}$$

$$g(\boldsymbol{t}) = \prod_{k=1}^{N}(1+\alpha_k\psi(p_k,\boldsymbol{t})\psi^*(q_k,\boldsymbol{t})) \tag{7}$$

where α_k $(k = 1, 2, \ldots, N)$ are complex constants,

$$\psi(p,\boldsymbol{t}) = p^{-\ell}(1-p)^{-m}\prod_{n'}^{n}(1+\delta_{n'}-p)^{-1}\psi(p)$$

$$\psi^*(q,\boldsymbol{t}) = q^{\ell}(1-q)^{m}\prod_{n'}^{n}(1+\delta_{n'}-q)\psi^*(q)$$

with

$$\prod_{n'}^{n} X_{n'} \equiv \begin{cases} \displaystyle\prod_{n'=1}^{n} X_{n'} & 1 \leqslant n \\ 1 & n = 0 \\ \displaystyle\prod_{n'=n+1}^{0} X_{n'}^{-1} & n \leqslant -1 \end{cases}$$

and $\psi(p)$, $\psi^*(q)$ are fermionic field operators which satisfy

$$\langle \mathrm{vac}|\psi(p_1)\psi(p_2),\ldots,\psi(p_r)\psi^*(q_r)\psi^*(q_{r-1}),\ldots,\psi^*(q_1)|\mathrm{vac}\rangle = \det\left(\frac{1}{p_i-q_j}\right)_{1\leqslant i,j\leqslant r}.$$

In order to relate the NDKP equation to the BBS, we impose a constraint on $\tau(\ell, m, n)$:

$$\tau(\ell, m, n) = \tau(\ell - M, m - 1, n). \tag{8}$$

Denoting $\sigma_s^n \equiv \tau(s-1, m=0, n)$, equation (5) turns into

$$(1+\delta_n)\sigma_{s-M}^{n-1}\sigma_{s+1}^{n} - \sigma_{s+1-M}^{n-1}\sigma_s^n - \delta_n\sigma_{s-M}^{n}\sigma_{s+1}^{n-1} = 0. \tag{9}$$

The N-soliton solution (7) is also a solution to equation (9) if it holds that

$$\left(\frac{q_k}{p_k}\right)^M\left(\frac{1-q_k}{1-p_k}\right) = 1 \tag{10}$$

for $k = 1, 2, \ldots, N$. It should be noted that, for a given p_k, there are M q_ks which satisfy equation (10) and $q_k \neq p_k$. We use this fact to construct explicit solutions to the BBS.

3. BBS as UD limit of the NDKP equation

We consider an infinite array of boxes in a line. The capacity of the nth $(-\infty < n < \infty)$ box is denoted by θ_n, which is a positive integer. We suppose that there are M kinds of balls distinguishable by an integer index j $(1 \leqslant j \leqslant M)$. The rule for time evolution of this BBS is the same as that given in section 1.

If $u_{n,j}^t$ denotes the number of balls with index j at time t in the nth box, the evolution rule given in the introduction is described as follows:

$$u_{n,j}^t = \min\left[\sum_{n'=-\infty}^{n-1} u_{n',j}^{t-1} - \sum_{n'=-\infty}^{n-1} u_{n',j}^{t}, \theta_n - \sum_{j'=1}^{j-1} u_{n,j'}^{t} - \sum_{j'=j}^{M} u_{n,j'}^{t-1}\right]. \tag{11}$$

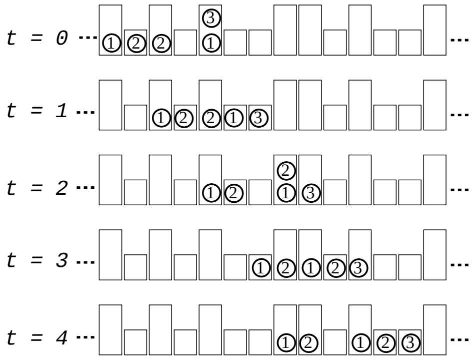

Figure 4. Two-soliton interaction of BBS with spatial dependence of box capacity.

We introduce a dependent variable Y_n^s $(s \equiv Mt + j)$ as

$$Y_n^s \equiv Y_n^{Mt+j} := \sum_{n'=-\infty}^{n} \left(\sum_{j'=j}^{M} u_{n',j'}^t + \sum_{t'=t+1}^{\infty} \sum_{j'=1}^{M} u_{n',j'}^{t'} \right).$$

From equation (11) and noticing the relation:

$$u_{n,j}^t = -Y_n^{s+1} + Y_n^s + Y_{n-1}^{s+1} - Y_{n-1}^s|_{s=Mt+j}$$

we have

$$Y_n^{s+1} + Y_{n-1}^{s-M} = \max[Y_n^s + Y_{n-1}^{s+1-M}, Y_{n-1}^{s+1} + Y_n^{s-M} - \theta_n]. \tag{12}$$

The form of equation (12) seems to suggest some connections of the BBS with the NDKP equation (9). In fact, equation (12) is obtained from equation (9) by the limiting procedure: UD. To see this, we introduce a small positive parameter ε. We put $\delta_n = \exp[-\theta_n/\varepsilon]$ in equation (9). Then a solution to equation (9) generically depends on the parameter $\varepsilon : \sigma_s^n \equiv \sigma_s^n(\varepsilon)$. Noticing the identity

$$\lim_{\varepsilon\to+0} \varepsilon \log(\exp[A/\varepsilon] + \exp[B/\varepsilon]) = \max[A, B] \qquad \text{for} \quad A, B \in \mathbb{R}$$

if the limit $\lim_{\varepsilon\to+0} \varepsilon \log \sigma_s^n(\varepsilon) \equiv \tilde{Y}_n^s$ exists, it is obvious that $Y_n^s = \tilde{Y}_n^s$ satisfies equation (12). Thus, once we find one parameter (ε) family of solutions $\sigma_s^n(\varepsilon)$, we can obtain a solution to the BBS. UD is this kind of method by which we can obtain a CA and its solutions at the same time through limiting procedures. Since the NDKP equation is essentially equivalent to the generating formula of KP hierarchy, we may regard the BBSs as a realization of ultra-discrete limit of KP hierarchy.

4. N-soliton solutions to the BBS

In this section, we construct explicit soliton solutions to the BBS with the aid of solutions to the NDKP equation.

First we consider the one-soliton solution. The one-soliton solution to the BBS is shown to have the form:

$$Y_n^{Mt+j} = \max\left[0,\ K_0 - tL - \sum_{i=1}^{j} \ell_i + \sum_{n'}^{n} \min[\theta_{n'}, L]\right] \tag{13}$$

where L is the length of soliton which corresponds to the number of all balls in the soliton, K_0 is an integer which is related to the phase of soliton, and ℓ_i $(i = 1, 2, \ldots, M)$ are the non-negative integers which correspond to the number of ith balls in the soliton. Thus it holds that $\sum_{i=1}^{M} \ell_i = L$. We shall give some details of its derivation, because multi-soliton solutions are obtained with similar arguments. To obtain (13), we take $g(\boldsymbol{t})$ in (7) as

$$g(\boldsymbol{t}) = \prod_{\ell=0}^{M-1} (1 + c_\ell(q_\ell)\psi(p, \boldsymbol{t})\psi^*(q_\ell, \boldsymbol{t})) \tag{14}$$

$$= 1 + \psi(p, \boldsymbol{t})\phi^*(p, \boldsymbol{t}) \tag{15}$$

$$\phi^*(p, \boldsymbol{t}) \equiv \sum_{\ell=0}^{M-1} c_\ell(q_\ell)\psi^*(q_\ell, \boldsymbol{t}) \tag{16}$$

where q_ℓ $(\ell = 0, 1, \ldots, M-1)$ are the roots of algebraic equation

$$\frac{x^M(1-x) - p^M(1-p)}{x-p} = 0 \qquad (x \neq p) \tag{17}$$

for a given real number p $(M/(M+1) < p < 1)$, and $c_\ell(p)$ $(0 \leqslant \ell \leqslant M-1)$ are complex coefficients which will be determined later. Since equation (17) has one real positive root except for p, we assume that q_0 is positive and we put $\gamma = q_0/p$. Then p and q_0 satisfy

$$p = \frac{1-\gamma^M}{1-\gamma^{M+1}} \tag{18}$$

$$1 - p = \gamma^M \left(\frac{1-\gamma}{1-\gamma^{M+1}}\right) \tag{19}$$

$$q_0 = \gamma \left(\frac{1-\gamma^M}{1-\gamma^{M+1}}\right). \tag{20}$$

The τ-function $\sigma_s^n (= \tau(\boldsymbol{t}))$ is given from equation (6) as

$$\sigma_n^s = 1 + \sum_{\ell=0}^{M-1} c_\ell(p) \frac{1}{p - q_\ell} \left(\frac{q_\ell}{p}\right)^s \prod_{n'}^{n} \left(\frac{1 - q_\ell/(1+\delta_{n'})}{1 - p/(1+\delta_{n'})}\right). \tag{21}$$

We introduce a small positive parameter ε and put $\gamma = \exp[-L/(M\varepsilon)]$ with an integer L. We also put

$$\tilde{c}_\ell(p) \equiv \frac{c_\ell(p)}{p - q_\ell}(1-q_\ell)^{T_0} \prod_{n'}^{N_0} (1 - q_\ell/(1+\delta_{n'})) \tag{22}$$

$$\chi_p(s) \equiv \sum_{\ell=0}^{M-1} \tilde{c}_\ell(p) \left(\frac{q_\ell}{p}\right)^s \tag{23}$$

where $T_0 = T_0(\varepsilon)$ and $N_0 = N_0(\varepsilon)$ are positive integers which satisfy $T_0 \simeq N_0 \simeq 1/\varepsilon$. Hence, $\lim_{\varepsilon\to+0} T_0 = \lim_{\varepsilon\to+0} N_0 = +\infty$.

We determine $c_\ell(p)$ $(\ell = 0, 1, 2, \ldots, M-1)$ by the following assumption for $\chi_p(j)$ $(j = 0, 1, 2, \ldots, M-1)$:

$$\begin{aligned}
&\chi_p(0) = \chi_0 \\
&\chi_p(1) = N_1 y^{\ell_1} \chi_p(0) \\
&\chi_p(2) = N_2 y^{\ell_2} \chi_p(1) \\
&\cdots \\
&\chi_p(M-1) = N_{M-1} y^{\ell_{M-1}} \chi_p(M-2).
\end{aligned} \tag{24}$$

Here χ_0 is a positive number which is related to the initial phase of soliton, $y = \exp[-1/\varepsilon]$, ℓ_j and $N_j = N_j(\varepsilon)$ ($j = 1, 2, \ldots, M-1$) are non-negative integers and positive numbers respectively. They are also supposed to satisfy

$$\begin{aligned}
&\ell_M \equiv L - \sum_{j=1}^{M-1} \ell_j \geqslant 0 \\
&\lim_{\varepsilon\to 0} \varepsilon \log N_j(\varepsilon) = 0 \\
&N_j y^{\ell_j} \leqslant \varepsilon^{N^*}
\end{aligned} \tag{25}$$

for a sufficiently large positive integer N^*. From (24) and (25), $c_\ell(p)$ ($\ell = 0, 1, \ldots, M-1$) are uniquely determined by the equation:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ q_0 & q_1 & \cdots & q_{M-1} \\ q_0^2 & q_1^2 & \cdots & q_{M-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_0^{M-1} & q_1^{M-1} & \cdots & q_{M-1}^{M-1} \end{pmatrix} \begin{pmatrix} \tilde{c}_0(p) \\ \tilde{c}_1(p) \\ \tilde{c}_2(p) \\ \vdots \\ \tilde{c}_{M-1}(p) \end{pmatrix} = \begin{pmatrix} \chi_p(0) \\ p\chi_p(1) \\ p^2\chi_p(2) \\ \vdots \\ p^{M-1}\chi_p(M-1) \end{pmatrix}. \tag{26}$$

Note that the determinant of the $M \times M$ matrix in the left-hand side of (26) is the Vandermond determinant: $\prod_{1\leqslant i<j\leqslant M-1}(q_j - q_i) \neq 0$.

Since

$$\begin{aligned}
\chi_p(s+M) &= \sum_{\ell=0}^{M-1} \tilde{c}_\ell(p) \left(\frac{q_\ell}{p}\right)^{s+M} \\
&= \sum_{\ell=0}^{M-1} \tilde{c}_\ell(p) \left(\frac{q_\ell}{p}\right)^{s} \left(\frac{1-p}{1-q_\ell}\right) \\
&= (1-p) \sum_{i=0}^{\infty} p^i \sum_{\ell=0}^{M-1} \tilde{c}_\ell(p) \left(\frac{q_\ell}{p}\right)^{s+i} \\
&= (1-p) \sum_{i=0}^{\infty} p^i \chi_p(s+i)
\end{aligned}$$

we have

$$\chi_p(s+M) = \sum_{i=0}^{M-1} \left(\sum_{\ell=0}^{\infty} (1-p)^{\ell+1} p^{M\ell} g_\ell(i) \right) p^i \chi_p(s+i) \tag{27}$$

where $g_0(i) = 1$, $g_1(i) = i + 1$ and

$$\begin{aligned}
g_\ell(i) &= \sum_{k_1=(\ell-1)M}^{(\ell-1)M+i} \sum_{k_2=(\ell-2)M}^{k_1} \cdots \sum_{k_\ell=0}^{k_{\ell-1}} 1 \\
&= \frac{(i+1)}{\ell!} \prod_{j=1}^{\ell-1} (\ell M + i + j + 1)
\end{aligned}$$

for $\ell \geqslant 2$. The ratio $g_{\ell+1}(i)/g_\ell(i)$ ($\ell \geqslant 1$) is calculated as

$$\begin{aligned}
\frac{g_{\ell+1}(i)}{g_\ell(i)} &= \frac{(\ell+1)(M+1)+i}{\ell+1} \prod_{k=1}^{\ell-1} \left(1 + \frac{M}{\ell M + i + k + 1}\right) \\
&< (M+1)\left(1 + \frac{1}{\ell}\right)^{\ell} < (M+1)\mathrm{e}.
\end{aligned}$$

Hence, if it holds that $(1-p)p^M < (M+1)^{-1}\mathrm{e}^{-1}$, we obtain

$$0 < \chi_p(s+M) \leqslant (1-p)\sum_{i=0}^{M-1}\left(1+(i+1)\frac{(1-p)p^M}{1-(1-p)p^M(M+1)\mathrm{e}}\right)\chi_p(s+i). \qquad (28)$$

Thus, from (28) and (25), we find that

$$\begin{aligned}&\chi_p(i) \geqslant \varepsilon^{-N^*}\chi_p(i+1) \qquad \text{for} \quad \forall i\\ &\chi_p(i) \geqslant C\exp[L/\varepsilon]\chi_p(i+M) \qquad \text{for} \quad \forall i \quad \text{and} \quad \exists C > 0.\end{aligned} \qquad (29)$$

Now we evaluate the τ functions σ_n^s and take its UD limit. From equation (21), we have

$$\begin{aligned}\sigma_n^{Mt+j} &= 1+\sum_{j=0}^{M-1}\tilde{c}_j(p)\left(\frac{q_j}{p}\right)^{Mt+j}(1-q_j)^{-T_0}\prod_{n'}^{n}\left(1-\frac{p}{1+\delta_{n'}}\right)^{-1}\prod_{n'=n+1}^{N_0}\left(1-\frac{q_j}{1+\delta_{n'}}\right)^{-1}\\ &= 1+\sum_{j=0}^{M-1}\tilde{c}_j(p)\left(\frac{q_j}{p}\right)^{j}\left(\frac{1-p}{1-q_j}\right)^{t}(1-q_j)^{-T_0}\\ &\quad\times\prod_{n'}^{n}\left(1-\frac{p}{1+\delta_{n'}}\right)^{-1}\prod_{n'=n+1}^{N_0}\left(1-\frac{q_j}{1+\delta_{n'}}\right)^{-1}.\end{aligned} \qquad (30)$$

For a moment, we assume that n and t are in the region: $|n| \leqslant N_0$ and $|t| \leqslant T_0$. Noticing that

$$\begin{aligned}(1-q_j)^{-T_0-t}\prod_{n'=n+1}^{N_0}\left(1-\frac{q_j}{1+\delta_{n'}}\right)^{-1} &= 1+\left(T_0+t+\sum_{n'=n+1}^{N_0}\left(\frac{1}{1+\delta_{n'}}\right)\right)q_j+\cdots\\ &\equiv 1+a_1\left(\frac{q_j}{p}\right)+a_2\left(\frac{q_j}{p}\right)^2+a_3\left(\frac{q_j}{p}\right)^3+\cdots\end{aligned}$$

we get

$$\sigma_n^{Mt+j} = 1+(1-p)^t\prod_{n'}^{n}\left(1-\frac{p}{1+\delta_{n'}}\right)^{-1}\sum_{i=0}^{\infty}a_i\chi_p(j+i) \qquad (31)$$

where $a_0 = 1$ and $a_{j+1}/a_j \sim \varepsilon^{-1}$. From (29), we have $0 < \sum_{i=1}^{\infty}a_i\chi_p(j+i) < \chi_p(j)$ for sufficiently small ε. Putting $\chi_0 = \exp[K_0/\varepsilon]$ and using the relations

$$\lim_{\varepsilon\to+0}\varepsilon\log(1-p) = -L$$

and

$$\lim_{\varepsilon\to+0}\varepsilon\log\left(1-\frac{p}{1+\delta_{n'}}\right)^{-1} = \min[L,\theta_n]$$

we obtain

$$\lim_{\varepsilon\to+0}\varepsilon\log\sigma_n^{Mt+j} = \max\left[0, K_0-tL-\sum_{i=1}^{j}\ell_i+\sum_{n'}^{n}\min[\theta_{n'},L]\right]. \qquad (32)$$

Since $\lim_{\varepsilon\to+0}N_0(\varepsilon) = \lim_{\varepsilon\to+0}T_0(\varepsilon) = +\infty$, these results are valid for any finite n and t. Thus we have shown that (13) is a solution to equation (12).

It may be interesting to see how $\varepsilon\log\sigma_n^s(\varepsilon) \equiv \tilde{Y}_n^s(\varepsilon)$ approaches to the right-hand side of (32). We define $\tilde{u}_n^s(\varepsilon) := -\tilde{Y}_n^{s+1}+\tilde{Y}_n^s+\tilde{Y}_{n-1}^{s+1}-\tilde{Y}_{n-1}^s$. By definition, we have $\tilde{u}_n^{Mt+j}(+0) = u_{n,j}^t$. Figure 5 shows $\tilde{u}_n^s(\varepsilon)$ given from (31) for various values of ε. They show fairly localizing behaviours and do not look like typical soliton solutions in 1 + 1 dimensions. In fact, as is seen from the construction (see (14)), the one-soliton solutions of this system should be regarded as degenerate M-soliton solutions.

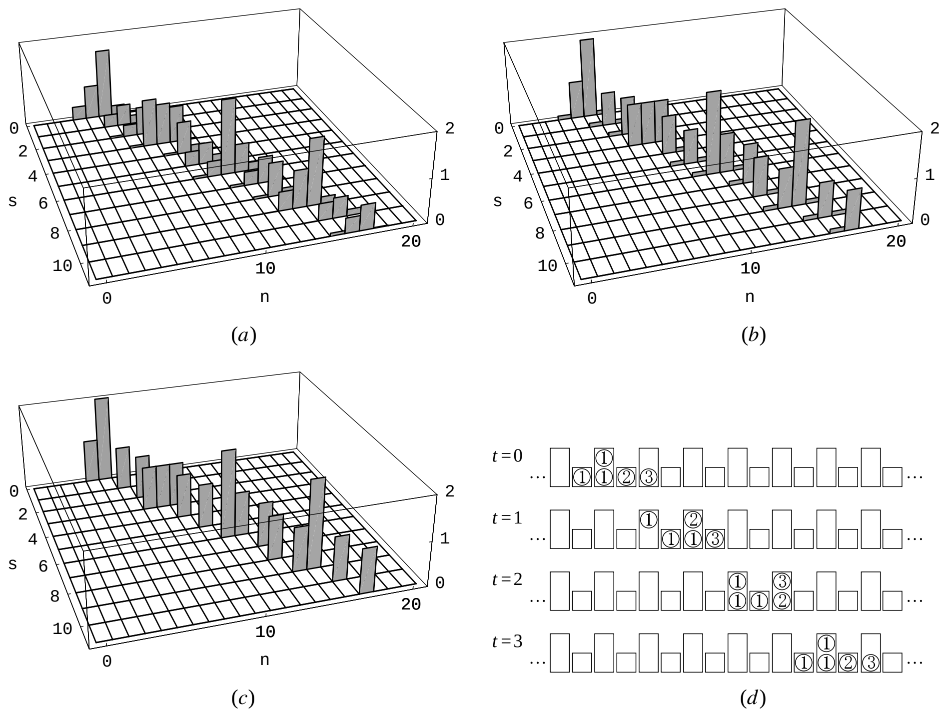

Figure 5. One-soliton solution to equation (12). (*a*) $y = \exp[-1/\varepsilon] = 1.0$; (*b*) $y = 0.01$; (*c*) $y = +0(\varepsilon = +0)$; (*d*) corresponding time evolution of the BBS.

Furthermore, in the limit $p \to 1 - 0$, we have $q_\ell \to \gamma \exp[2\pi\sqrt{-1}\ell/M]$ and $\chi_p(\ell)$ becomes ℓth Fourier component of a function $\Sigma_n(s) = \gamma^{-t}\sigma_n^{Mt+j}$ which is a periodic function with respect to $s \equiv Mt + j$ with period M. Hence, in the UD limit, we can construct any shape of periodic function with period M by suitably choosing $\chi_p(j)(0 \leqslant j \leqslant M-1)$ in (24), though it is no longer expressed as a solution of a BBS. For finite ε, these type of solutions exhibit solitonical behaviours with complicated inner structures.

We turn to the construction of multi-soliton solutions of the BBS. From the above arguments, we see that the field operators $\psi(p)$ and $\phi^*(p)$ are essentially determined by L, ℓ_j $(j = 1, 2, \ldots, M)$ and K_0. Therefore we denote these operators by

$$\psi(p) = \psi(L : \varepsilon) \qquad \phi^*(p) = \phi^*(L; \{\ell_j\}; K_0 : \varepsilon). \tag{33}$$

For two-soliton solutions, we take

$$g(\boldsymbol{t}) = (1 + \psi(p_1, \boldsymbol{t})\phi^*(p_1, \boldsymbol{t}))(1 + \psi(p_2, \boldsymbol{t})\phi^*(p_2, \boldsymbol{t})) \tag{34}$$

where

$$\psi(p_i) = \psi(L^{(i)} : \varepsilon) \qquad \phi^*(p_i) = \phi^*(L^{(i)}; \{\ell_j^{(i)}\}; K_0^{(i)} : \varepsilon) \qquad (i = 1, 2). \tag{35}$$

We also assume $L^{(1)} \geqslant L^{(2)}$ and $\ell_j^{(1)} \geqslant \ell_j^{(2)}$ $(j = 1, 2, \ldots, M)$. As we shall see below, the latter condition turns out to be a natural constraint for soliton solutions. Using notation as above, we have

$$\begin{aligned}\sigma_n^{Mt+j} &= \langle \mathrm{vac}|(1 + \psi(p_1, \boldsymbol{t})\phi^*(p_1, \boldsymbol{t}))(1 + \psi(p_2, \boldsymbol{t})\phi^*(p_2, \boldsymbol{t}))|\mathrm{vac}\rangle \\ &= 1 + \langle \mathrm{vac}|\psi(p_1, \boldsymbol{t})\phi^*(p_1, \boldsymbol{t})|\mathrm{vac}\rangle + \langle \mathrm{vac}|\psi(p_2, \boldsymbol{t})\phi^*(p_2, \boldsymbol{t})|\mathrm{vac}\rangle \\ &\quad + \langle \mathrm{vac}|\psi(p_1, \boldsymbol{t})\phi^*(p_1, \boldsymbol{t})\psi(p_2, \boldsymbol{t})\phi^*(p_2, \boldsymbol{t})|\mathrm{vac}\rangle.\end{aligned} \tag{36}$$

The second and third terms are calculated in the same way as above. The fourth term is calculated as

$$
\begin{aligned}
&\langle \mathrm{vac}|\psi(p_1,\boldsymbol{t})\phi^*(p_1,\boldsymbol{t})\psi(p_2,\boldsymbol{t})\phi^*(p_2,\boldsymbol{t})|\mathrm{vac}\rangle \\
&\qquad = \sum_{j=0}^{M-1}\sum_{j'=0}^{M-1}\tilde{c}_j(p_1)\tilde{c}_{j'}(p_2)\left(\frac{(p_1-p_2)(q_{j'}^{(2)}-q_j^{(1)})}{(p_1-q_{j'}^{(2)})(p_2-q_j^{(1)})}\right) \\
&\qquad\quad \times\prod_{i=1,2}\left(\frac{q_j^{(i)}}{p_i}\right)^j\left(\frac{1-p_i}{1-q_j^{(i)}}\right)^t(1-q_j^{(i)})^{-T_0} \\
&\qquad\quad \times\prod_{n'}^{n}\left(1-\frac{p_i}{1+\delta_{n'}}\right)^{-1}\prod_{n'=n+1}^{N_0}\left(1-\frac{q_j^{(i)}}{1+\delta_{n'}}\right)^{-1}.
\end{aligned}
\tag{37}
$$

We define $\chi_{p_i}(s)$ by

$$
\chi_{p_i}(s)\equiv\sum_{\ell=0}^{M-1}\tilde{c}_\ell(p_i)\left(\frac{q_\ell^{(i)}}{p_i}\right)^s \qquad (i=1,2)
\tag{38}
$$

and suppose

$$
\begin{aligned}
&\chi_{p_i}(0)=\chi_0^{(i)} \\
&\chi_{p_i}(1)=N_1^{(i)}y^{\ell_1^{(i)}}\chi_{p_i}(0) \\
&\chi_{p_i}(2)=N_2^{(i)}y^{\ell_2^{(i)}}\chi_{p_i}(1) \\
&\cdots \\
&\chi_{p_i}(M-1)=N_{M-1}^{(i)}y^{\ell_{M-1}^{(i)}}\chi_{p_i}(M-2)
\end{aligned}
\tag{39}
$$

where positive numbers $N_j^{(i)}$ satisfy the similar inequalities to (25). Note that $\ell_j^{(1)}\geqslant\ell_j^{(2)}$ $(j=1,2,\ldots,M)$ and it is always possible to choose $N_j^{(i)}$ such that

$$
\frac{\chi_{p_2}(j+1)}{\chi_{p_2}(j)}\gg\frac{\chi_{p_1}(j+1)}{\chi_{p_1}(j)}.
\tag{40}
$$

Then (37) is expanded as

$$
\begin{aligned}
&\frac{(p_1-p_2)(1-p_1)^t(1-p_2)^t}{p_1p_2}\prod_{n'}^{n}\left(1-\frac{p_1}{1+\delta_{n'}}\right)^{-1}\left(1-\frac{p_2}{1+\delta_{n'}}\right)^{-1} \\
&\qquad \times\sum_{i=0}^{\infty}\sum_{i'=0}^{\infty}(a_{i,i'}\chi_{p_1}(j+i)\chi_{p_2}(j+1+i')-b_{i,i'}\chi_{p_2}(j+i)\chi_{p_1}(j+1+i'))
\end{aligned}
$$

where $a_{0,0}=b_{0,0}=1$ and, from (29), we evaluate

$$
\begin{aligned}
&\chi_{p_1}(j)\chi_{p_2}(j+1)>\sum_{i=0}^{\infty}\sum_{\substack{i'=0\\ i+i'\neq0}}^{\infty}a_{i,i'}\chi_{p_1}(j+i)\chi_{p_2}(j+1+i') \\
&\chi_{p_2}(j)\chi_{p_1}(j+1)>\sum_{i=0}^{\infty}\sum_{\substack{i'=0\\ i+i'\neq0}}^{\infty}b_{i,i'}\chi_{p_2}(j+i)\chi_{p_1}(j+1+i').
\end{aligned}
$$

Then, using (40), we find

$$
\begin{aligned}
&\lim_{\varepsilon\to+0}\varepsilon\log\sigma_n^s=\max[0,K^{(1)}(s,n),K^{(2)}(s,n)K^{(1)}(s,n)+K^{(2)}(s,n)+A(Mt+j)] \\
&K^{(i)}(n,Mt+j)\equiv K_0^{(i)}-tL^{(i)}-\sum_{j'=1}^{j}\ell_{j'}^{(i)}+\sum_{n'}^{n}\min[\theta_{n'},L^{(i)}] \quad (i=1,2)
\end{aligned}
\tag{41}
$$

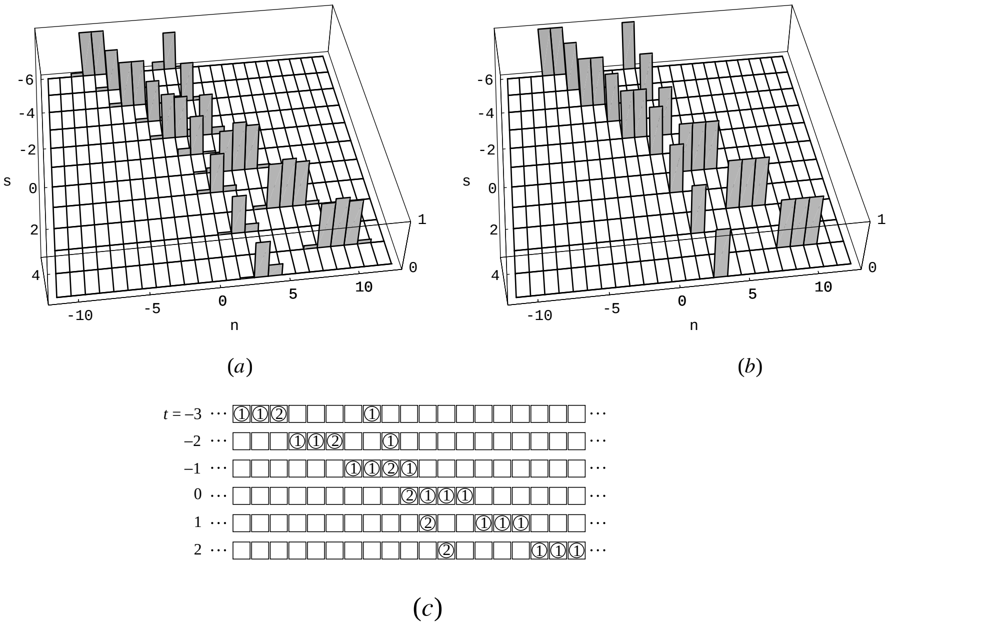

Figure 6. Two-soliton solution to equation (12). (*a*) $y \equiv \exp[-1/\varepsilon] = 1.0^{-4}$; (*b*) $y = +0(\varepsilon \to +0)$; (*c*) corresponding time evolution of the BBS.

$$A(Mt + j) \equiv L^{(2)} + \ell_{j+1}^{(2)} \qquad (\text{modulo } M). \tag{42}$$

This gives a two-soliton solution. We show an example of a two-soliton solution to equation (12) for finite ε and the corresponding BBS in figure 6.

The integer $\ell_j^{(1)}$ $(1 \leqslant j \leqslant M)$ corresponds to the number of jth balls in the larger soliton at $t \to -\infty$, and $\ell_j^{(2)}$ corresponds to that of the smaller soliton at $t \to +\infty$. Since the order of balls with the same number (same species) does not change in time evolution, the balls in the smaller soliton at $t \to +\infty$ must be included in the larger soliton at $t \to -\infty$. Therefore the condition $\ell_j^{(1)} \geqslant \ell_j^{(2)}$ must hold for any two-soliton solutions. We should also note that there are several freedoms to choose the phase $A(s)$ in taking the UD limit. However, we conjecture that the other choices give essentially the same time evolution patterns for the BBS.

N-soliton solutions are obtained in the same way and we only show the results. We take

$$g(\boldsymbol{t}) = \prod_{i=1}^{N} (1 + \psi(p_i, \boldsymbol{t})\phi^*(p_i, \boldsymbol{t})) \tag{43}$$

where

$$\psi(p_i) = \psi(L^{(i)} : \varepsilon) \qquad \phi^*(p_i) = \phi^*(L^{(i)}; \{\ell_j^{(i)}\}; K_0^{(i)} : \varepsilon) \qquad (i = 1, 2, \ldots, N). \tag{44}$$

We suppose that

$$L^{(1)} \geqslant L^{(2)} \geqslant \cdots \geqslant L^{(N)}$$

and

$$\ell_j^{(1)} \geqslant \ell_j^{(2)} \geqslant \cdots \geqslant \ell_j^{(N)} \qquad (j = 1, 2, \ldots, N).$$

The latter condition is a natural constraint of N-soliton solutions of the BBS as is the case for two-soliton solutions. Then N-soliton solutions are given by

$$Y_n^s = \max_{\vec{\mu}} \left[\sum_{i=1}^{N} \mu_i K^{(i)}(s,n) - A(\vec{\mu}; s) \right]. \tag{45}$$

Here $\vec{\mu} \equiv (\mu_1, \mu_2, \ldots, \mu_N)$ $(\mu_i = 0, 1)$ and $\max_{\vec{\mu}}[\cdots]$ denotes the maximum value among 2^N values which are obtained by putting $\mu_i = 0$ or 1 for $i = 1, 2, \ldots, N$, and

$$K^{(i)}(Mt + j, n) \equiv K_0^{(i)} - tL^{(i)} - \sum_{j'=1}^{j} \ell_{j'}^{(i)} + \sum_{n'}^{n} \min[\theta_{n'}, L^{(i)}]$$

with an arbitrary integer $K_0^{(i)}$. In the case

$$\begin{aligned} \mu_i &= 1 \qquad \text{for} \quad i = i_1, i_2, \ldots, i_p \\ \mu_i &= 0 \qquad \text{otherwise} \end{aligned}$$

the phase factor $A(\vec{\mu}; s)$ is given by

$$A(\vec{\mu}; s) \equiv \sum_{k=1}^{p} (k-1) L^{(i_k)} + \sum_{k=1}^{p} (X^{(i_k)}(s + k - 1) - X^{(i_k)}(s))$$

where $X^{(i)}(Mt + j) \equiv tL^{(i)} + \sum_{j'=1}^{j} \ell_{j'}$.

5. Conclusion

We have investigated CAs which are realized by the movements of balls in an array of an infinite number of boxes. We showed that the BBSs are obtained by UD of the NDKP equation and that the spatial dependence of the capacity of each box corresponds to a nonautonomous variable of the NDKP equation. The explicit expressions of the N-soliton solutions to the BBSs are presented with the aid of some peculiar soliton solutions of the NDKP equation.

Although our solutions seem to cover all the soliton solutions to the BBSs, we have not found the proof yet. We may need another approach which was effective in the case of box capacity one [18, 19], which is a future problem. In BBSs, there is also another freedom: capacity of carrier [8]. Extension to the system including this freedom is also another future problem.

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References

[1] Wolfram S (ed) 1986 *Theory and Applications of Cellular Automata* (Singapore: World Scientific)

[2] Berlekamp E R, Conway J H and Guy R K 1984 *Winning Ways for your Mathematical Plays* vol 2 (New York: Academic)

[3] Park J K, Steiglitz K and Thurston W P 1986 *Physica* D **19** 423

[4] Fokas A S, Papadopoulou E, Saridakis Y and Ablowits M J 1989 *Stud. Appl. Math.* **81** 153

[5] Bruschi M, Santini P M and Ragnisco O 1992 *Phys. Lett.* A **169** 151
Bruschi M and Santini P M 1994 *Physica* D **70** 185

[6] Takahashi D and Satsuma J 1990 *J. Phys. Soc. Japan* **59** 3514

[7] Takahashi D 1992 *Nonlinear Evolution Equations and Dynamical Systems* ed M Boiti *et al* (Singapore: World Scientific) p 245

Takahashi D 1993 *NOLTA'93: Proc. Int. Symp. on Nonlinear Theory and its Applications* p 555

[8] Takahashi D and Matsukidaira J 1997 *J. Phys. A: Math. Gen.* **30** L733

[9] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 *Phys. Rev. Lett.* **76** 3247

[10] Matsukidaira J, Satsuma J, Takahashi D, Tokihiro T and Torii M 1997 *Phys. Lett.* A **255** 287

[11] Willox R, Tokihiro T and Satsuma J 1997 *J. Math. Phys.* **38** 6455

[12] Nagai A, Tokihiro T, Satsuma J, Willox R and Kajiwara K 1997 *Phys. Lett.* A **234** 301

[13] Miwa T, Jimbo M and Date E 1993 *Soliton No Suuri* (Tokyo: Iwanami) (in Japanese)

[14] Jimbo M and Miwa T 1983 *Publ. RIMS (Kyoto University)* **19** 943

[15] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785

[16] Miwa T 1982 *Proc. Japan Acad.* A **58** 9

[17] Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Japan* **51** 4116

[18] Nagai A, Takahashi D and Tokihiro T 1999 *Phys. Lett.* A **255** 265

[19] Tokihiro T, Nagai A and Satsuma J 1999 *Inverse Problems* **15** 1639–62